5.5 Use Inequalities in a Triangle

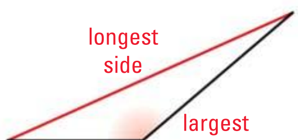
Before You found what combinations of angles are possible in a triangle.
Now You will find possible side lengths of a triangle.
Why? So you can find possible distances, as in Ex. 39.

Key Vocabulary
- **side opposite,** *p. 241*
- **inequality,** *p. 876*

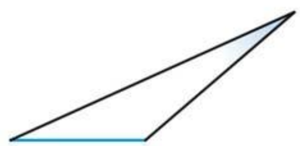
EXAMPLE 1 Relate side length and angle measure

Draw an obtuse scalene triangle. Find the largest angle and longest side and mark them in red. Find the smallest angle and shortest side and mark them in blue. What do you notice?

Solution



The longest side and largest angle are opposite each other.



The shortest side and smallest angle are opposite each other.

The relationships in Example 1 are true for all triangles as stated in the two theorems below. These relationships can help you to decide whether a particular arrangement of side lengths and angle measures in a triangle may be possible.

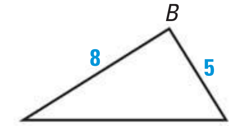
AVOID ERRORS
Be careful not to confuse the symbol ∠ meaning *angle* with the symbol < meaning *is less than*. Notice that the bottom edge of the angle symbol is horizontal.

THEOREMS *For Your Notebook*

THEOREM 5.10

If one side of a triangle is longer than another side, then the angle opposite the longer side is larger than the angle opposite the shorter side.

Proof: p. 329

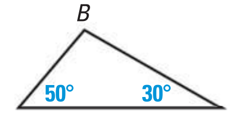


$AB > BC$, so $m\angle C > m\angle A$.

THEOREM 5.11

If one angle of a triangle is larger than another angle, then the side opposite the larger angle is longer than the side opposite the smaller angle.

Proof: Ex. 24, p. 340



$m\angle A > m\angle C$, so $BC > AB$.

EXAMPLE 2 Standardized Test Practice

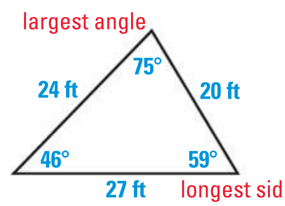
STAGE PROP You are constructing a stage prop that shows a large triangular mountain. The bottom edge of the mountain is about 27 feet long, the left slope is about 24 feet long, and the right slope is about 20 feet long. You are told that one of the angles is about 46° and one is about 59°. What is the angle measure of the peak of the mountain?

Ⓐ 46° Ⓑ 59° Ⓒ 75° Ⓓ 85°

ELIMINATE CHOICES
You can eliminate choice D because a triangle with a 46° angle and a 59° angle cannot have an 85° angle. The sum of the three angles in a triangle must be 180°, but the sum of 46, 59, and 85 is 190, not 180.

Solution

Draw a diagram and label the side lengths. The peak angle is opposite the longest side so, by Theorem 5.10, the peak angle is the largest angle.

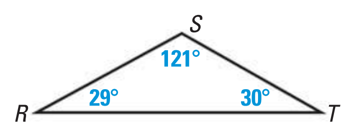


The angle measures sum to 180°, so the third angle measure is $180° - (46° + 59°) = 75°$. You can now label the angle measures in your diagram.

▶ The greatest angle measure is 75°, so the correct answer is C. Ⓐ Ⓑ Ⓒ Ⓓ

GUIDED PRACTICE for Examples 1 and 2

1. List the sides of $\triangle RST$ in order from shortest to longest.

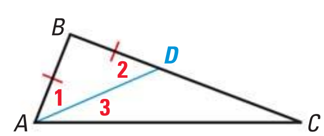


2. Another stage prop is a right triangle with sides that are 6, 8, and 10 feet long and angles of 90°, about 37°, and about 53°. Sketch and label a diagram with the shortest side on the bottom and the right angle at the left.

PROOF Theorem 5.10

GIVEN ▶ $BC > AB$

PROVE ▶ $m\angle BAC > m\angle C$



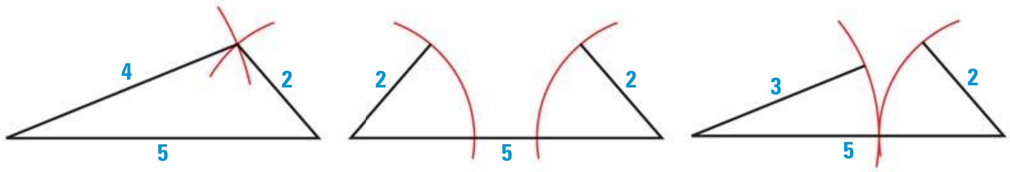
Locate a point *D* on $\overline{BC}$ such that $DB = BA$. Then draw $\overline{AD}$. In the isosceles triangle $\triangle ABD$, $\angle 1 \cong \angle 2$.

Because $m\angle BAC = m\angle 1 + m\angle 3$, it follows that $m\angle BAC > m\angle 1$. Substituting $m\angle 2$ for $m\angle 1$ produces $m\angle BAC > m\angle 2$.

By the Exterior Angle Theorem, $m\angle 2 = m\angle 3 + m\angle C$, so it follows that $m\angle 2 > m\angle C$ (see Exercise 27, page 332). Finally, because $m\angle BAC > m\angle 2$ and $m\angle 2 > m\angle C$, you can conclude that $m\angle BAC > m\angle C$.

THE TRIANGLE INEQUALITY Not every group of three segments can be used to form a triangle. The lengths of the segments must fit a certain relationship.

For example, three attempted triangle constructions for sides with given lengths are shown below. Only the first set of side lengths forms a triangle.



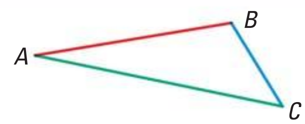
If you start with the longest side and attach the other two sides at its endpoints, you can see that the other two sides are not long enough to form a triangle in the second and third figures. This leads to the *Triangle Inequality Theorem.*

Animated Geometry at classzone.com

THEOREM — *For Your Notebook*

THEOREM 5.12 Triangle Inequality Theorem

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.



$AB + BC > AC$ $\quad AC + BC > AB$ $\quad AB + AC > BC$

Proof: Ex. 47, p. 334

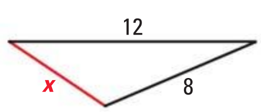
EXAMPLE 3 Find possible side lengths

ALGEBRA A triangle has one side of length 12 and another of length 8. Describe the possible lengths of the third side.

Solution

Let x represent the length of the third side. Draw diagrams to help visualize the small and large values of x. Then use the Triangle Inequality Theorem to write and solve inequalities.

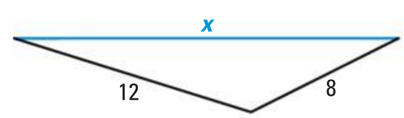
Small values of *x*



$x + 8 > 12$

$x > 4$

Large values of *x*



$8 + 12 > x$

$20 > x$, or $x < 20$

▶ The length of the third side must be greater than 4 and less than 20.

USE SYMBOLS
You can combine the two inequalities, $x > 4$ and $x < 20$, to write the compound inequality $4 < x < 20$. This can be read as *x is between 4 and 20.*

GUIDED PRACTICE for Example 3

3. A triangle has one side of 11 inches and another of 15 inches. *Describe* the possible lengths of the third side.

5.5 EXERCISES

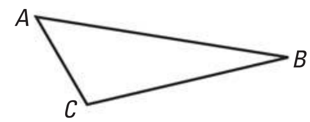
HOMEWORK KEY

◯ = **WORKED-OUT SOLUTIONS** on p. WS1 for Exs. 9, 17, and 39

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 12, 20, 30, 39, and 45

SKILL PRACTICE

1. VOCABULARY Use the diagram at the right. For each angle, name the side that is *opposite* that angle.



2. ★ WRITING How can you tell from the angle measures of a triangle which side of the triangle is the longest? the shortest?

EXAMPLE 1 on p. 328 for Exs. 3–5

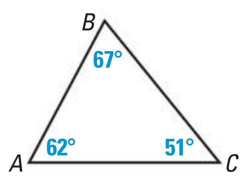
MEASURING Use a ruler and protractor to draw the given type of triangle. Mark the largest angle and longest side in red and the smallest angle and shortest side in blue. What do you notice?

3. Acute scalene **4.** Right scalene **5.** Obtuse isosceles

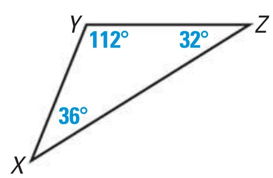
EXAMPLE 2 on p. 329 for Exs. 6–15

WRITING MEASUREMENTS IN ORDER List the sides and the angles in order from smallest to largest.

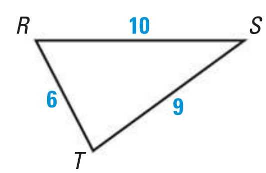
6.



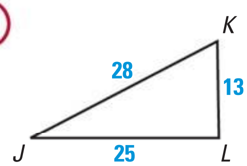
7.



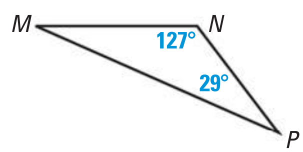
8.



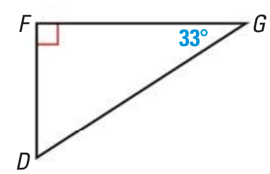
9.



10.



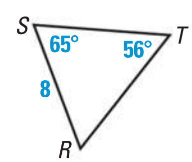
11.



12. ★ MULTIPLE CHOICE In △*RST*, which is a possible side length for *ST*?

Ⓐ 7 Ⓑ 8

Ⓒ 9 Ⓓ Cannot be determined



DRAWING TRIANGLES Sketch and label the triangle described.

13. Side lengths: about 3 m, 7 m, and 9 m, with longest side on the bottom
Angle measures: 16°, 41°, and 123°, with smallest angle at the left

14. Side lengths: 37 ft, 35 ft, and 12 ft, with shortest side at the right
Angle measures: about 71°, about 19°, and 90°, with right angle at the top

15. Side lengths: 11 in., 13 in., and 14 in., with middle-length side at the left
Two angle measures: about 48° and 71°, with largest angle at the top

EXAMPLE 3 on p. 330 for Exs. 16–26

IDENTIFYING POSSIBLE TRIANGLES Is it possible to construct a triangle with the given side lengths? If not, *explain* why not.

16. 6, 7, 11 **17.** 3, 6, 9 **18.** 28, 34, 39 **19.** 35, 120, 125

20. ★ **MULTIPLE CHOICE** Which group of side lengths can be used to construct a triangle?

(A) 3 yd, 4 ft, 5 yd (B) 3 yd, 5 ft, 8 ft

(C) 11 in., 16 in., 27 in. (D) 2 ft, 11 in., 12 in.

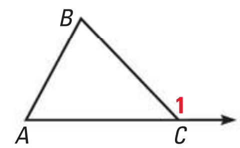
POSSIBLE SIDE LENGTHS ***Describe*** **the possible lengths of the third side of the triangle given the lengths of the other two sides.**

21. 5 inches, 12 inches
22. 3 meters, 4 meters
23. 12 feet, 18 feet
24. 10 yards, 23 yards
25. 2 feet, 40 inches
26. 25 meters, 25 meters

27. **EXTERIOR ANGLE INEQUALITY** Another triangle inequality relationship is given by the Exterior Inequality Theorem. It states:

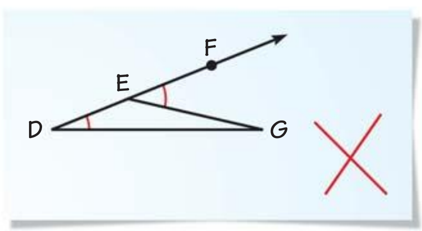
The measure of an exterior angle of a triangle is greater than the measure of either of the nonadjacent interior angles.

Use a relationship from Chapter 4 to *explain* how you know that $m\angle 1 > m\angle A$ and $m\angle 1 > m\angle B$ in $\triangle ABC$ with exterior angle $\angle 1$.

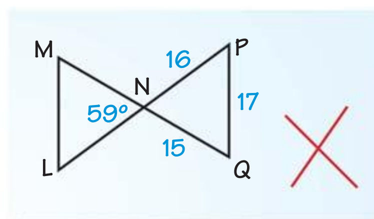


ERROR ANALYSIS **Use Theorems 5.10–5.12 and the theorem in Exercise 27 to** ***explain*** **why the diagram must be incorrect.**

28.



29.



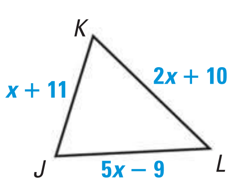
30. ★ **SHORT RESPONSE** *Explain* why the hypotenuse of a right triangle must always be longer than either leg.

ORDERING MEASURES **Is it possible to build a triangle using the given side lengths? If so, order the angles measures of the triangle from least to greatest.**

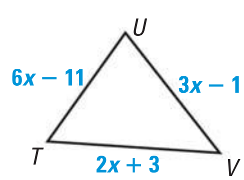
31. $PQ = \sqrt{58}$, $QR = 2\sqrt{13}$, $PR = 5\sqrt{2}$
32. $ST = \sqrt{29}$, $TU = 2\sqrt{17}$, $SU = 13.9$

xy ALGEBRA ***Describe*** **the possible values of *x*.**

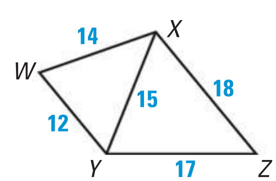
33.



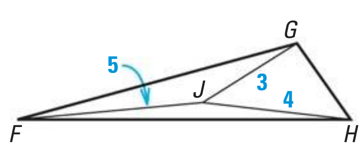
34.



35. **USING SIDE LENGTHS** Use the diagram at the right. Suppose $\overline{XY}$ bisects $\angle WYZ$. List all six angles of $\triangle XYZ$ and $\triangle WXY$ in order from smallest to largest. *Explain* your reasoning.



36. **CHALLENGE** The perimeter of $\triangle HGF$ must be between what two integers? *Explain* your reasoning.



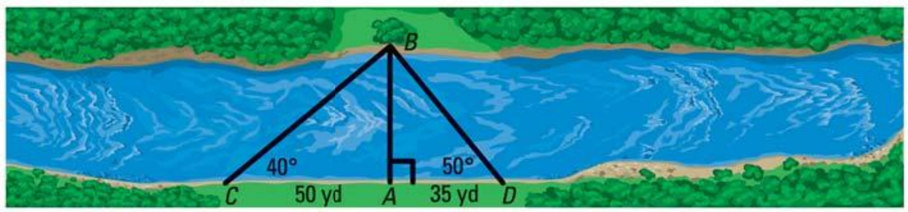
○ = WORKED-OUT SOLUTIONS on p. WS1 ★ = STANDARDIZED TEST PRACTICE

PROBLEM SOLVING

37. **TRAY TABLE** In the tray table shown, $\overline{PQ} \cong \overline{PR}$ and $QR < PQ$. Write two inequalities about the angles in $\triangle PQR$. What other angle relationship do you know?

@HomeTutor for problem solving help at classzone.com

38. **INDIRECT MEASUREMENT** You can estimate the width of the river at point A by taking several sightings to the tree across the river at point B. The diagram shows the results for locations C and D along the riverbank. Using $\triangle BCA$ and $\triangle BDA$, what can you conclude about AB, the width of the river at point A? What could you do if you wanted a closer estimate?



@HomeTutor for problem solving help at classzone.com

EXAMPLE 3
on p. 330
for Ex. 39

39. ★ **EXTENDED RESPONSE** You are planning a vacation to Montana. You want to visit the destinations shown in the map.

a. A brochure states that the distance between Granite Peak and Fort Peck Lake is 1080 kilometers. *Explain* how you know that this distance is a misprint.

b. Could the distance from Granite Peak to Fort Peck Lake be 40 kilometers? *Explain*.

c. Write two inequalities to represent the range of possible distances from Granite Peak to Fort Peck Lake.

d. What can you say about the distance between Granite Peak and Fort Peck Lake if you know that $m\angle 2 < m\angle 1$ and $m\angle 2 < m\angle 3$?

FORMING TRIANGLES **In Exercises 40–43, you are given a 24 centimeter piece of string. You want to form a triangle out of the string so that the length of each side is a whole number. Draw figures accurately.**

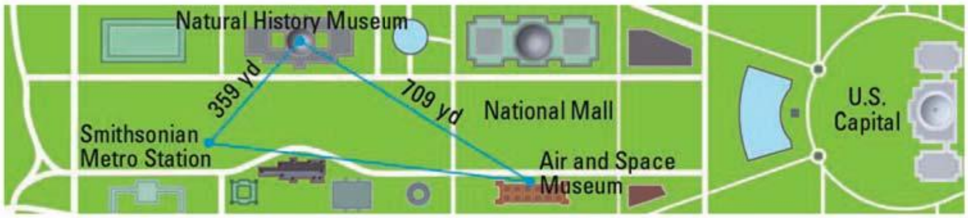
40. Can you decide if three side lengths form a triangle without checking all three inequalities shown for Theorem 5.12? If so, *describe* your shortcut.

41. Draw four possible isosceles triangles and label each side length. Tell whether each of the triangles you formed is *acute*, *right*, or *obtuse*.

42. Draw three possible scalene triangles and label each side length. Try to form at least one scalene acute triangle and one scalene obtuse triangle.

43. List three combinations of side lengths that will not produce triangles.

44. **SIGHTSEEING** You get off the Washington, D.C., subway system at the Smithsonian Metro station. First you visit the Museum of Natural History. Then you go to the Air and Space Museum. You record the distances you walk on your map as shown. *Describe* the range of possible distances you might have to walk to get back to the Smithsonian Metro station.



45. ★ **SHORT RESPONSE** Your house is 2 miles from the library. The library is $\frac{3}{4}$ mile from the grocery store. What do you know about the distance from your house to the grocery store? *Explain*. Include the special case when the three locations are all in a straight line.

46. **ISOSCELES TRIANGLES** For what combinations of angle measures in an isosceles triangle are the congruent sides shorter than the base of the triangle? longer than the base of the triangle?

47. **PROVING THEOREM 5.12** Prove the Triangle Inequality Theorem.

GIVEN ▶ $\triangle ABC$

PROVE ▶ (1) $AB + BC > AC$
(2) $AC + BC > AB$
(3) $AB + AC > BC$

Plan for Proof One side, say BC, is longer than or at least as long as each of the other sides. Then (1) and (2) are true. To prove (3), extend $\overline{AC}$ to D so that $\overline{AB} \cong \overline{AD}$ and use Theorem 5.11 to show that $DC > BC$.

48. **CHALLENGE** Prove the following statements.

a. The length of any one median of a triangle is less than half the perimeter of the triangle.

b. The sum of the lengths of the three medians of a triangle is greater than half the perimeter of the triangle.

MIXED REVIEW

PREVIEW
Prepare for Lesson 5.6 in Exs. 49–50.

In Exercises 49 and 50, write the if-then form, the converse, the inverse, and the contrapositive of the given statement. *(p. 79)*

49. A redwood is a large tree.

50. $5x - 2 = 18$, because $x = 4$.

51. A triangle has vertices $A(22, 21)$, $B(0, 0)$, and $C(22, 2)$. Graph $\triangle ABC$ and classify it by its sides. Then determine if it is a right triangle. *(p. 217)*

Graph figure *LMNP* with vertices $L(-4, 6)$, $M(4, 8)$, $N(2, 2)$, and $P(-4, 0)$. Then draw its image after the transformation. *(p. 272)*

52. $(x, y) \rightarrow (x + 3, y - 4)$

53. $(x, y) \rightarrow (x, -y)$

54. $(x, y) \rightarrow (-x, y)$